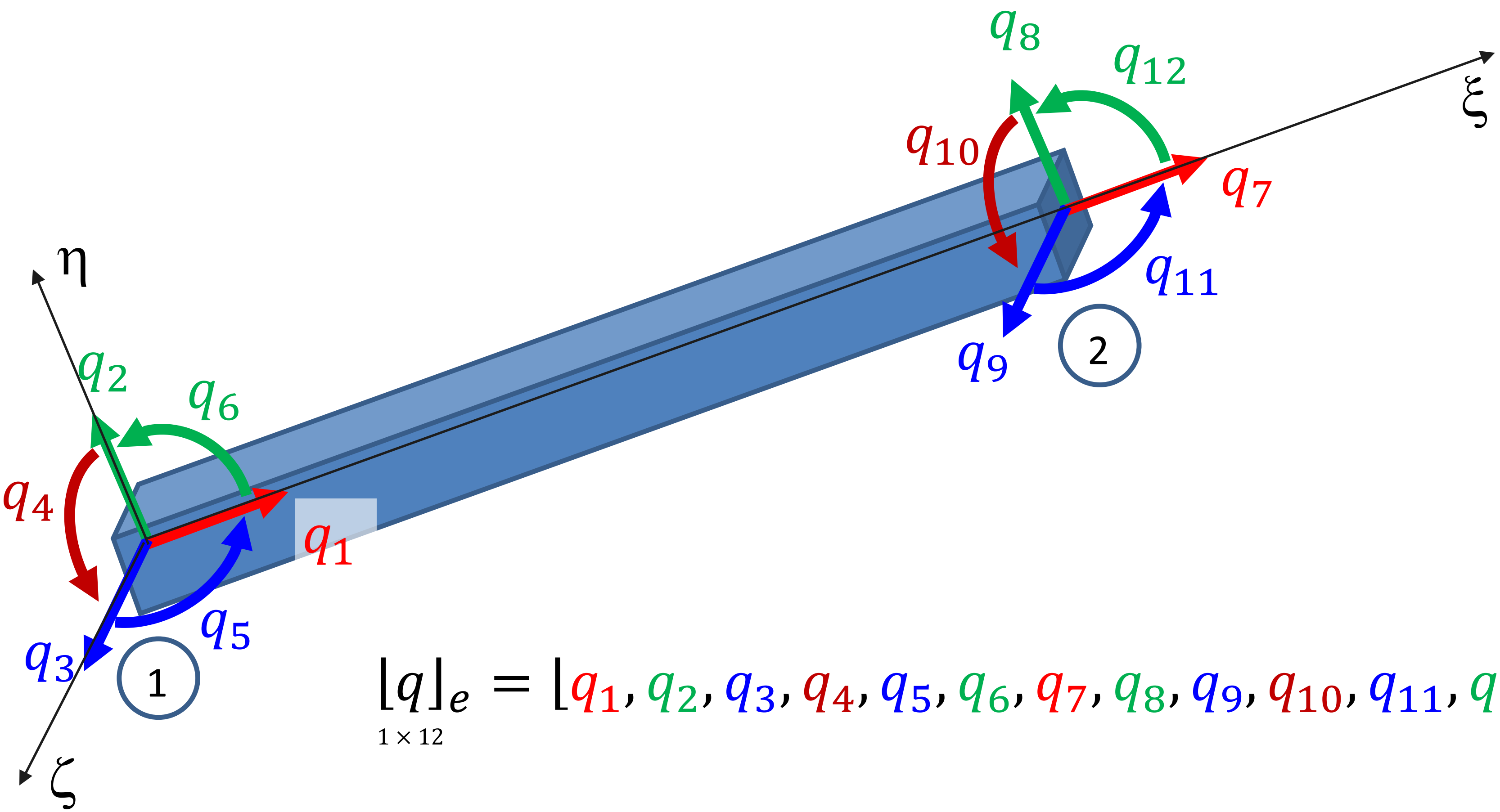
Politechnika Warszawska | Wydział Mechaniczny Energetyki i Lotnictwa | Zakład Wytrzymałości Materiałów i Konstrukcji

Metoda elementów skończonych (MES1)

Wykład 11A. Element ramy 3D

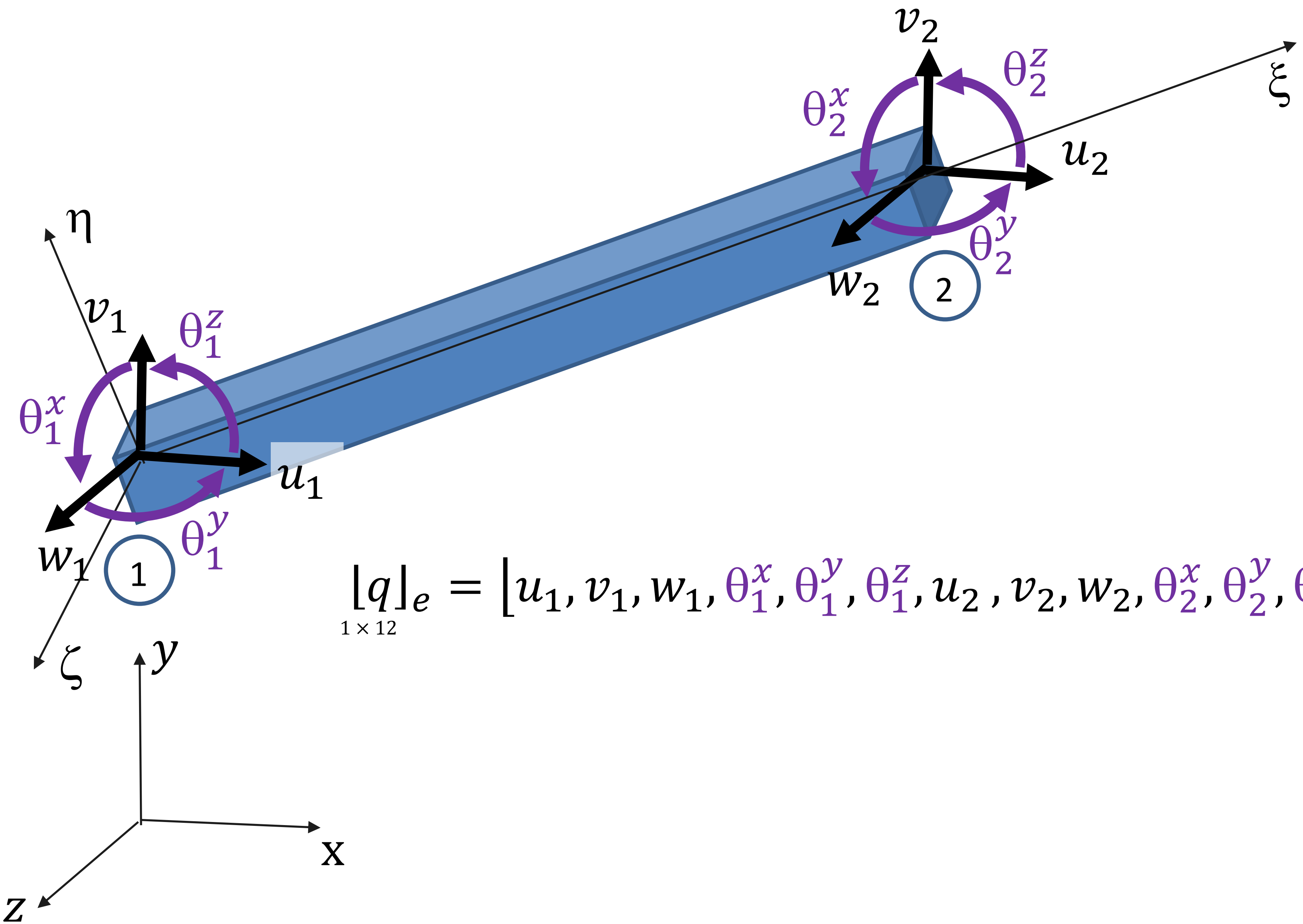
05.2022

Element ramy w lokalnym układzie współrzędnych (ξ, η, ζ)



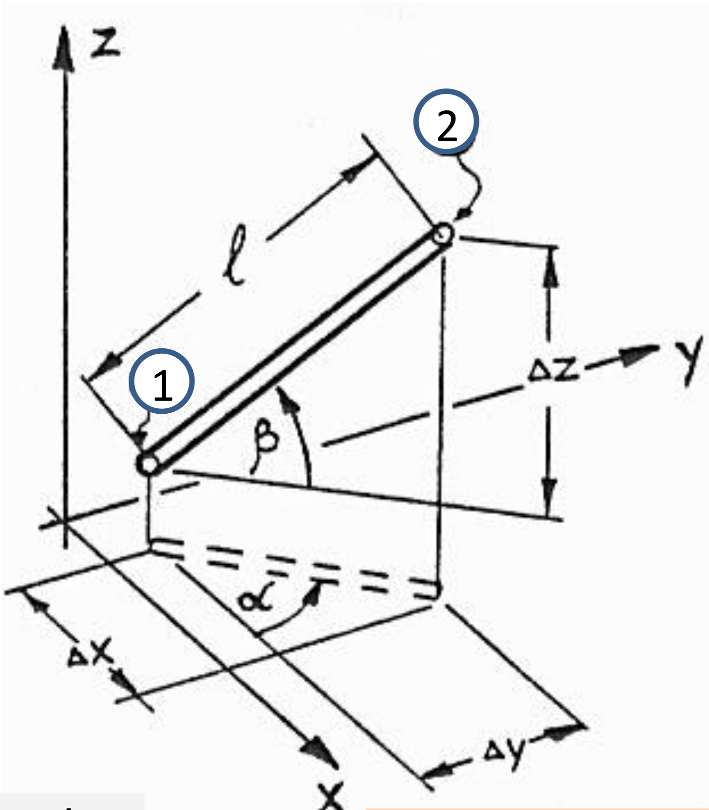
$$\underset{1\times 12}{\lfloor q \rfloor_e} = \lfloor q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}, q_{11}, q_{12} \rfloor$$

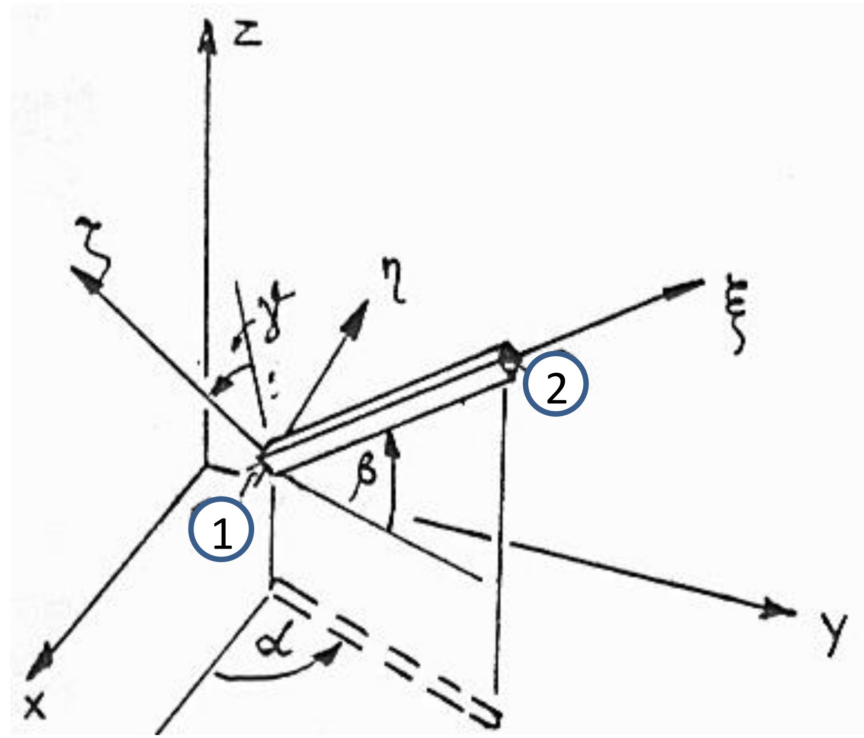
Element ramy w lokalnym globalnym współrzędnych (x, y, z)



$$\underset{1\times 12}{\lfloor q \rfloor_e} = \lfloor u_1, v_1, w_1, \theta_1^x, \theta_1^y, \theta_1^z, u_2, v_2, w_2, \theta_2^x, \theta_2^y, \theta_2^z \rfloor$$

Transformacja z układu globalnego (x, y, z) do lokalnego (ξ, η, ζ)





$s\alpha = sin\alpha$

$c\alpha = cos\alpha$

$s\alpha = sin\beta$

$c\beta = cos\beta$

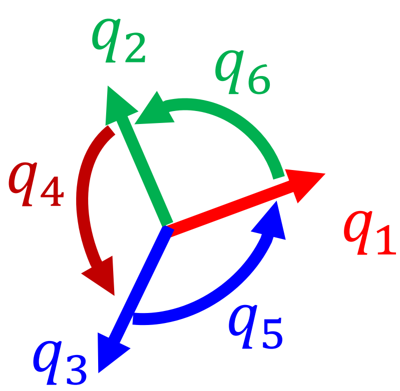
$s\gamma = sin\ \gamma$

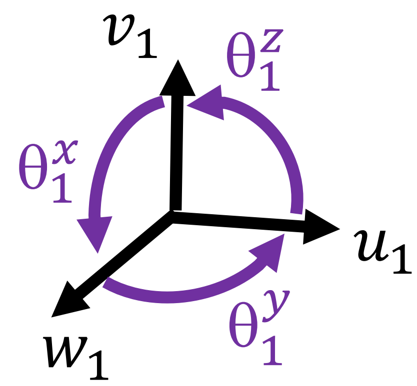
$c\gamma = cos\ \gamma$

$$\lfloor \xi, \eta, \zeta \rfloor^T = [T] \lfloor x, y, z \rfloor^T$$

$$[T] = \begin{bmatrix} c\alpha \cdot c\beta & s\alpha \cdot c\beta & -s\beta \\ -(c\alpha \cdot s\beta \cdot s\gamma + s\alpha \cdot c\gamma) & -(s\alpha \cdot s\beta \cdot s\gamma - c\alpha \cdot c\gamma) & c\beta \cdot s\gamma \\ -(c\alpha \cdot s\beta \cdot c\gamma - s\alpha \cdot s\gamma) & -(s\alpha \cdot s\beta \cdot c\gamma + c\alpha \cdot s\gamma) & c\beta \cdot c\gamma \end{bmatrix}$$

Transformacja stopni swobody z układu globalnego (x, y, z) do lokalnego (ξ, η, ζ)



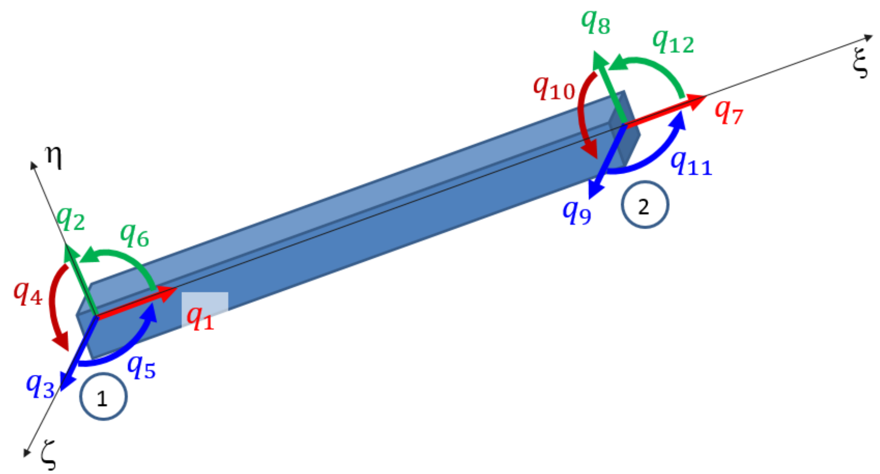


$$\{q\}_e = \begin{bmatrix} [T] & \\ & [T] \end{bmatrix} \{q_g\}_e = [T_r]\{q_g\}_e$$

$s\alpha = sin\alpha$
$c\alpha = cos\alpha$
$s\alpha = sin\beta$
$c\beta = cos\beta$
$s\gamma = sin\,\gamma$
$c\gamma = cos\,\gamma$

$$[T] = \begin{bmatrix} c\alpha \cdot c\beta & s\alpha \cdot c\beta & -s\beta \\ -(c\alpha \cdot s\beta \cdot s\gamma + s\alpha \cdot c\gamma) & -(s\alpha \cdot s\beta \cdot s\gamma - c\alpha \cdot c\gamma) & c\beta \cdot s\gamma \\ -(c\alpha \cdot s\beta \cdot c\gamma - s\alpha \cdot s\gamma) & -(s\alpha \cdot s\beta \cdot c\gamma + c\alpha \cdot s\gamma) & c\beta \cdot c\gamma \end{bmatrix}$$

Składowe macierze sztywności elementu ramy



$$[k_N]= \begin{bmatrix} \frac{AE}{l} & \frac{-AE}{l} \\ \frac{-AE}{l} & \frac{AE}{l} \end{bmatrix}$$

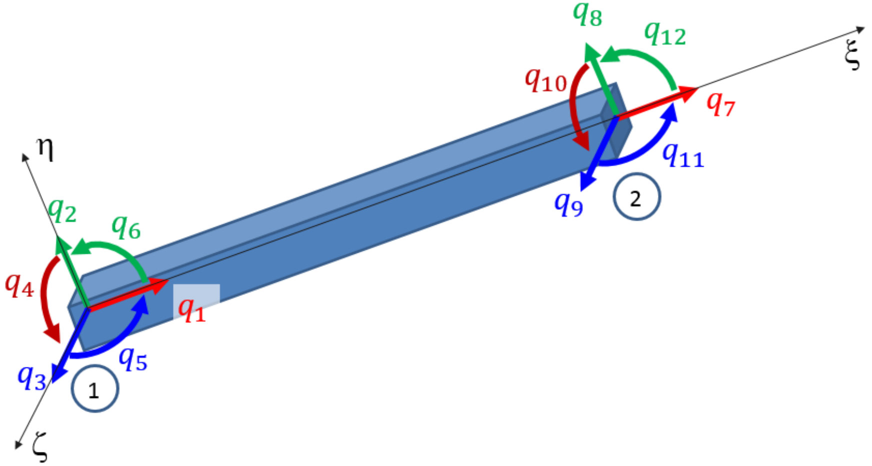
Rozciąganie dla stopni swobody $[q_1, , q_7]$

$$[k_S]= \begin{bmatrix} \frac{GJ_S}{l} & \frac{-GJ_S}{l} \\ \frac{-GJ_S}{l} & \frac{GJ_S}{l} \end{bmatrix}$$

Skręcanie dla stopni swobody $[q_4, q_{10}]$

Składowe macierze sztywności elementu ramy

$$\left[k_{M_g}\eta\right]=\begin{bmatrix} \frac{12EJ_\eta}{l^3} & \frac{-6EJ_\eta}{l^2} & \frac{-12EJ_\eta}{l^3} & \frac{-6EJ_\eta}{l^2} \\ \frac{-6EJ_\eta}{l^2} & \frac{4EJ_\eta}{l} & \frac{6EJ_\eta}{l^2} & \frac{2EJ_\eta}{l} \\ \frac{-12EJ_\eta}{l^3} & \frac{6EJ_\eta}{l^2} & \frac{12EJ_\eta}{l^3} & \frac{6EJ_\eta}{l^2} \\ \frac{-6EJ_\eta}{l^2} & \frac{2EJ_\eta}{l} & \frac{6EJ_\eta}{l^2} & \frac{4EJ_\eta}{l} \end{bmatrix}$$

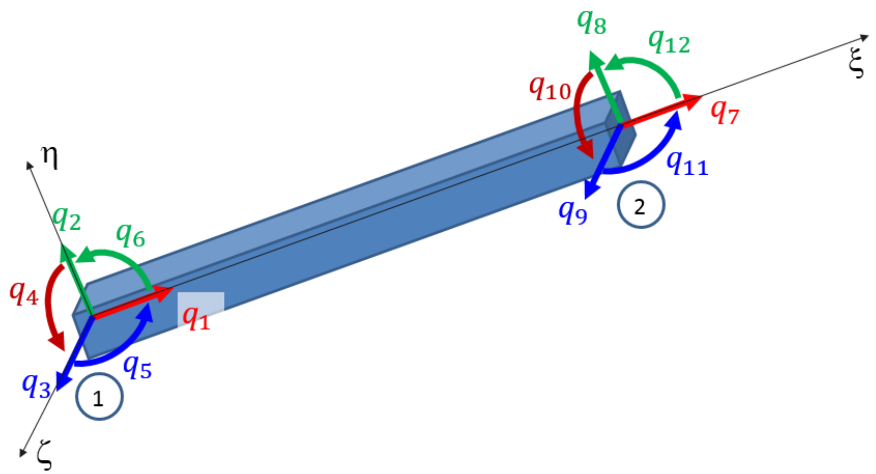


zginanie względem osi η dla stopni swobody $[q_3, q_5, q_9, q_{11}]$

$$\left[k_{M_g}\zeta\right]=\begin{bmatrix} \frac{12EJ_\zeta}{l^3} & \frac{6EJ_\zeta}{l^2} & \frac{-12EJ_\zeta}{l^3} & \frac{6EJ_\zeta}{l^2} \\ \frac{6EJ_\zeta}{l^2} & \frac{4EJ_\zeta}{l} & \frac{-6EJ_\zeta}{l^2} & \frac{2EJ_\zeta}{l} \\ \frac{-12EJ_\zeta}{l^3} & \frac{-6EJ_\zeta}{l^2} & \frac{12EJ_\zeta}{l^3} & \frac{-6EJ_\zeta}{l^2} \\ \frac{6EJ_\zeta}{l^2} & \frac{2EJ_\zeta}{l} & \frac{-6EJ_\zeta}{l^2} & \frac{4EJ_\zeta}{l} \end{bmatrix}$$

zginanie względem osi ζ dla stopni swobody $[q_2, q_6, q_8, q_{12}]$

Energia sprężysta elementu ramy



Energia sprężysta elementu:

$$U_e = \frac{1}{2} \lfloor q \rfloor_e [k]_e \{q\}_e = \frac{1}{2} \lfloor q_g \rfloor_e [T_r]^T [k]_e [T_r] \{q_g\}_e ,$$

$$U_e = \frac{1}{2} \lfloor q_g \rfloor_e \left[k^g \right]_e \{q_g\}_e ,$$

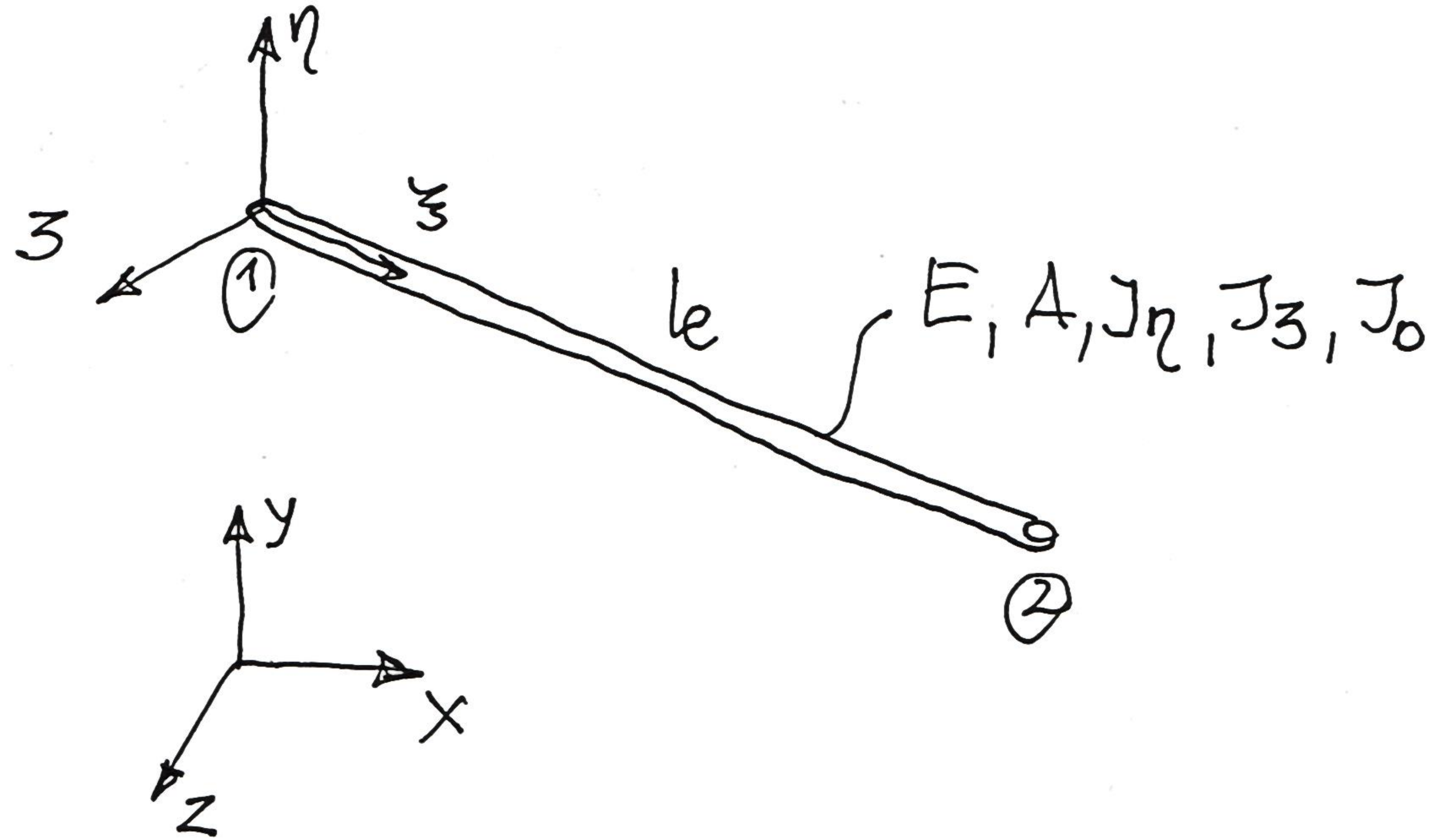
Macierz sztywności elementu:

$$\boxed{\left[k^g \right]_e = [T_r]^T [k]_e [T_r]}$$

Macierz sztywności elementu ramy w układzie lokalnym (ξ, η, ζ)

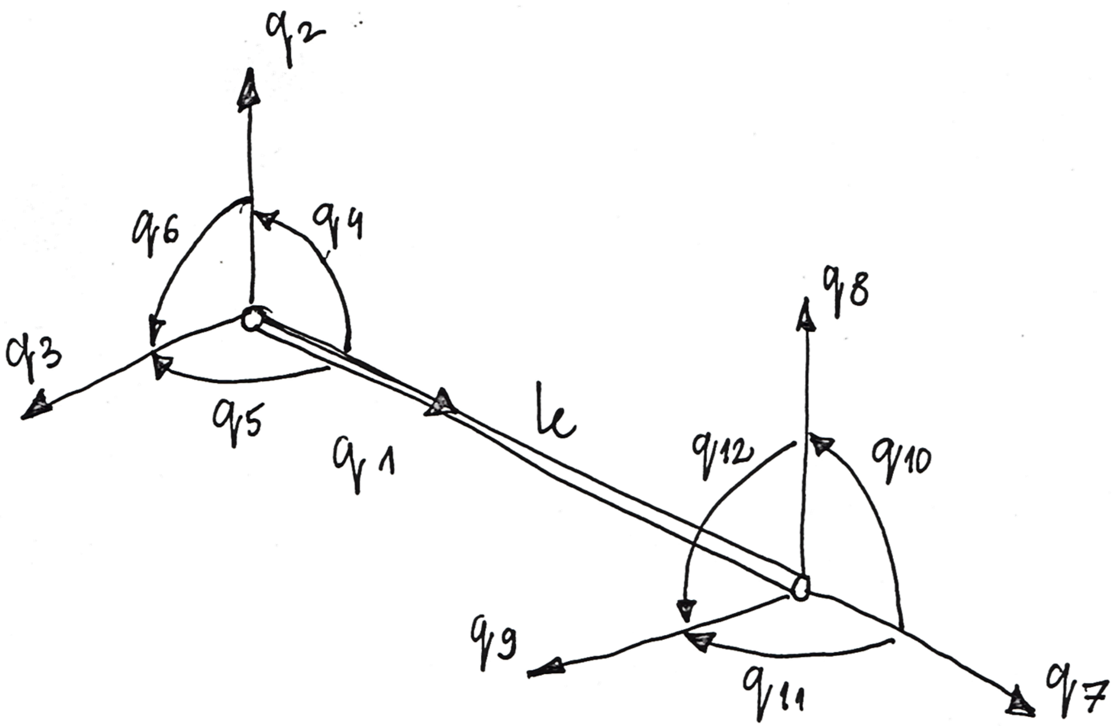
$$
[k_e]=\begin{bmatrix}
\frac{AE}{l} & & & & & & \frac{-AE}{l} & & & & & \\
& \frac{12EJ_\zeta}{l^3} & & & & \frac{6EJ_\zeta}{l^2} & & \frac{-12EJ_\zeta}{l^3} & & & & \frac{6EJ_\zeta}{l^2} \\
& & \frac{12EJ_\eta}{l^3} & & \frac{-6EJ_\eta}{l^2} & & & & \frac{-12EJ_\eta}{l^3} & & \frac{-6EJ_\eta}{l^2} & \\
& & & \frac{GJ_s}{l} & & & & & & \frac{-GJ_s}{l} & & \\
& & \frac{-6EJ_\eta}{l^2} & & \frac{4EJ_\eta}{l} & & & & \frac{6EJ_\eta}{l^2} & & \frac{2EJ_\eta}{l} & \\
& \frac{6EJ_\zeta}{l^2} & & & & \frac{4EJ_\zeta}{l} & & \frac{-6EJ_\zeta}{l^2} & & & & \frac{2EJ_\zeta}{l} \\
\frac{-AE}{l} & & & & & & \frac{AE}{l} & & & & & \\
& \frac{-12EJ_\zeta}{l^3} & & & & \frac{-6EJ_\zeta}{l^2} & & \frac{12EJ_\zeta}{l^3} & & & & \frac{-6EJ_\zeta}{l^2} \\
& & \frac{-12EJ_\eta}{l^3} & & \frac{6EJ_\eta}{l^2} & & & & \frac{12EJ_\eta}{l^3} & & \frac{6EJ_\eta}{l^2} & \\
& & & \frac{-GJ_s}{l} & & & & & & \frac{GJ_s}{l} & & \\
& & \frac{-6EJ_\eta}{l^2} & & \frac{2EJ_\eta}{l} & & & & \frac{6EJ_\eta}{l^2} & & \frac{4EJ_\eta}{l} & \\
& \frac{6EJ_\zeta}{l^2} & & & & \frac{2EJ_\zeta}{l} & & \frac{-6EJ_\zeta}{l^2} & & & & \frac{4EJ_\zeta}{l}
\end{bmatrix}
$$

3D FRAME ELEMENT



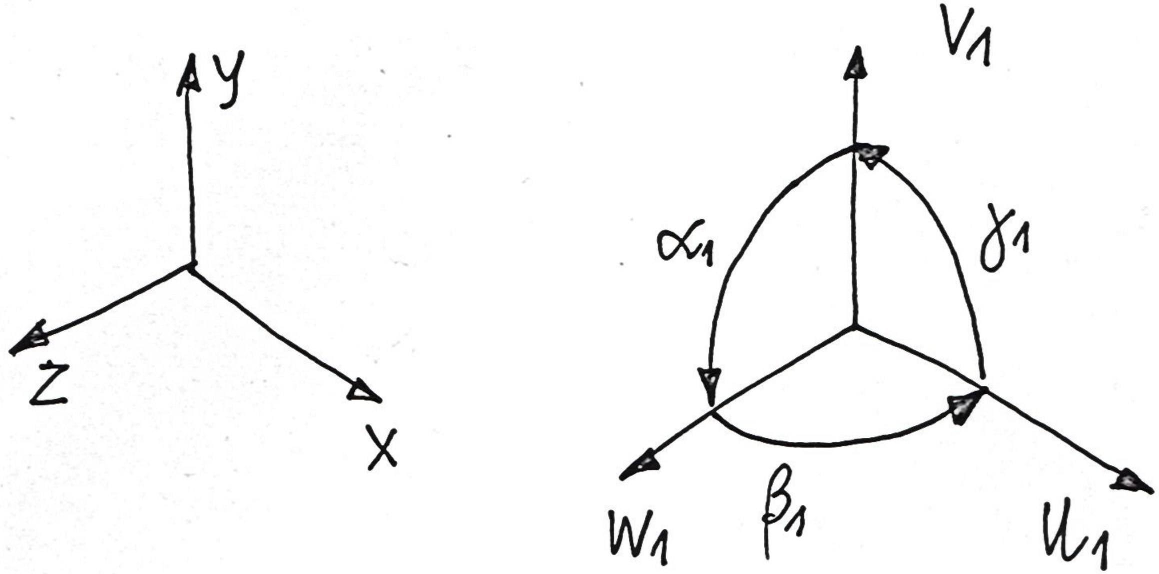
LOCAL PARAMETERS IN THE COORDINATE SYSTEM $\xi\eta\zeta$

$$\underset{1\times 12}{\lfloor q \rfloor_e} = \lfloor q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}, q_{11}, q_{12} \rfloor$$



Local parameters in the coordinate system xyz

$$\lfloor q_g \rfloor_{e \atop 1\times 12} = \lfloor u_1, v_1, w_1, \alpha_1, \beta_1, \gamma_1, u_2, v_2, w_2, \alpha_2, \beta_2, \gamma_2 \rfloor$$



$$U_e = \frac{1}{2} \underset{1\times 12}{\lfloor q \rfloor_e} \underset{12\times 12}{[k]_e} \underset{12\times 1}{\{q\}_e} \qquad \text{where :}$$

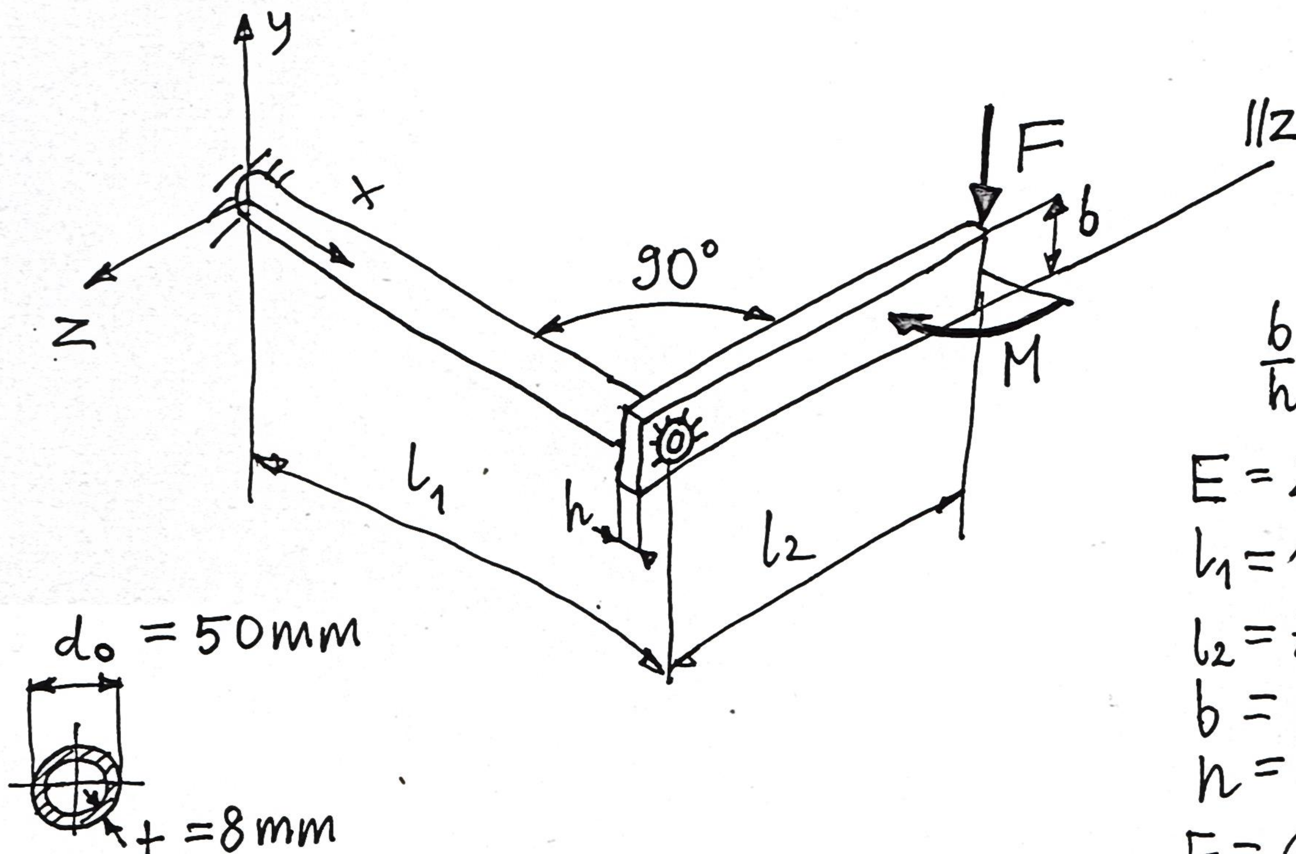
$$[k]_e = \begin{bmatrix}
a & & & & & & -a & & & & & \\
 & b & & d & & & & -b & & d & & \\
 & & c & & e & & & & -c & & e & \\
 & d & & 2r & & & & -d & & r & & \\
 & & e & & 2s & & & & -e & & s & \\
 & & & & & t & & & & & & -t \\
-a & & & & & & a & & & & & \\
 & -b & & -d & & & & b & & -d & & \\
 & & -c & & -e & & & & c & & -e & \\
 & d & & r & & & & -d & & 2r & & \\
 & & e & & s & & & & -e & & 2s & \\
 & & & & & -t & & & & & & t
\end{bmatrix}$$

$$a = \frac{EA}{L}, \quad b = \frac{12EJ_3}{l_e^3}, \quad c = \frac{12EJ_2}{l_e^3}, \quad d = \frac{6EJ_3}{l_e^2},$$

$$e = \frac{6EJ_2}{l_e^2}, \quad r = \frac{2EJ_3}{l_e}, \quad s = \frac{2EJ_2}{l_e}, \quad t = \frac{G \cdot J_0}{l_e}$$

$$U_e = \underset{1\times12}{\frac{1}{2}\lfloor q_g \rfloor_e} \cdot \underbrace{\underset{12\times12}{[T_f]^T} \cdot \underset{12\times12}{[k]_e} \cdot \underset{12\times12}{[T_f]}}_{\underset{12\times12}{[kg]_e}} \cdot \underset{12\times1}{\{q_g\}_e}$$

EXAMPLE : BUILD A FE MODEL USING 3D FRAME ELEMENTS. FIND UNKNOWN DISPLACEMENTS, STRESSES AND REACTIONS.



$\frac{b}{h} = 2$

$E = 2 \cdot 10^5$ MPa

$l_1 = 1200$ mm

$l_2 = 750$ mm

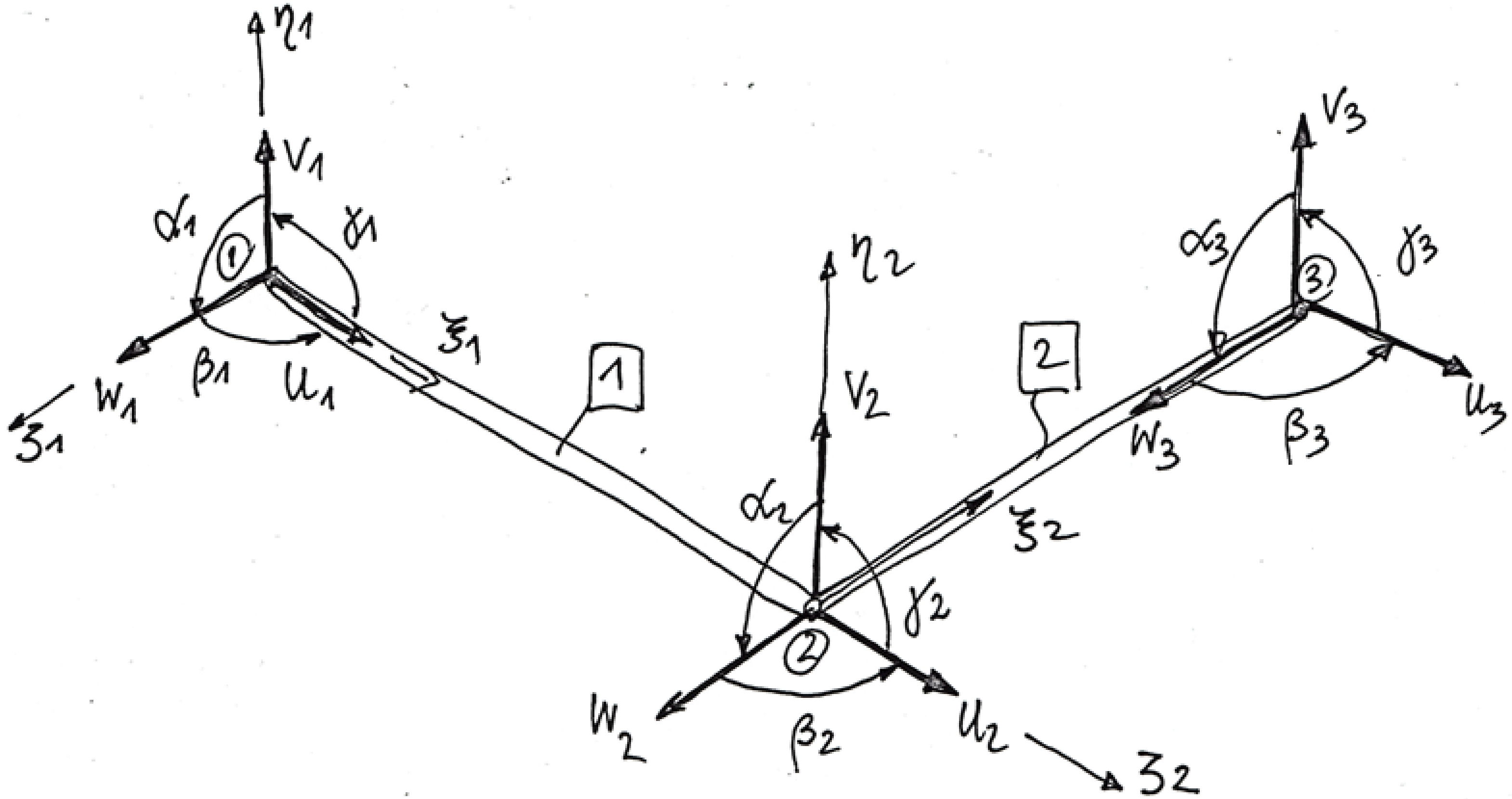
$b = 60$ mm

$h = 30$ mm

$F = 1000$ N

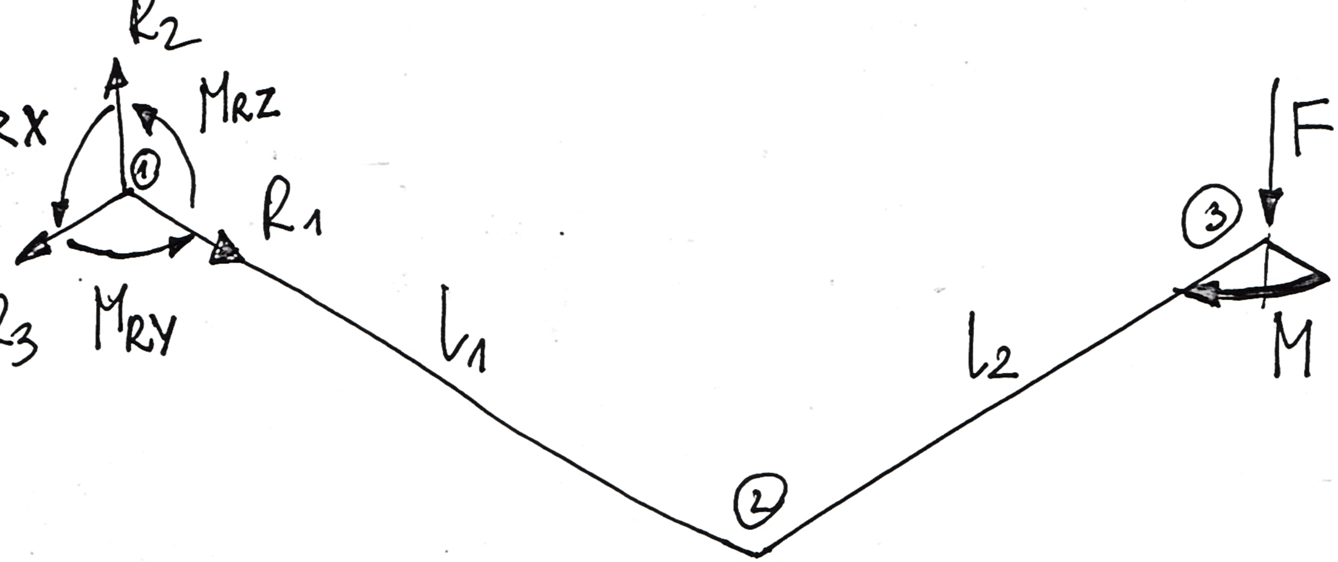
$M = 1000000$ Nmm

Global parameters in the coordinate system xyz



$$\underset{1\times 18}{\lfloor q \rfloor} = \lfloor u_1, v_1, w_1, \alpha_1, \beta_1, \gamma_1, u_2, v_2, w_2, \alpha_2, \beta_2, \gamma_2, u_3, v_3, w_3, \alpha_3, \beta_3, \gamma_3 \rfloor$$

Global load vector



$$\lfloor F \rfloor_{1\times 18} = \lfloor R_1, R_2, R_3, M_{RX}, M_{RY}, M_{RZ}, 0, 0, 0, 0, 0, 0, 0, -F, 0, 0, -M, 0 \rfloor$$

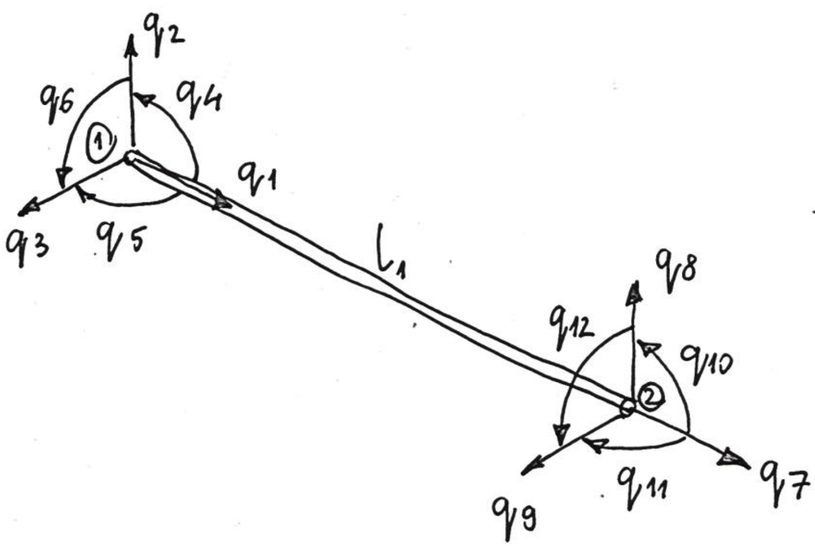
ELEMENT $\boxed{1}$:

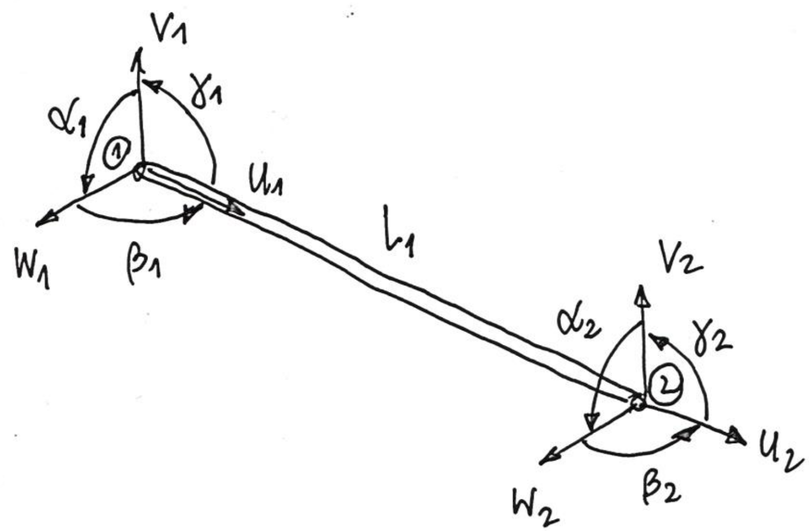
$$\underset{1\times 12}{\lfloor q \rfloor_1} = \lfloor q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}, q_{11}, q_{12} \rfloor$$

$$\underset{1\times 12}{\lfloor q_g \rfloor_1} = \lfloor u_1, v_1, w_1, \alpha_1, \beta_1, \gamma_1, u_2, v_2, w_2, \alpha_2, \beta_2, \gamma_2 \rfloor$$

$$q_1 = u_1, \quad q_2 = v_1, \quad q_3 = w_1, \quad q_4 = \gamma_1, \quad q_5 = -\beta_1, \quad q_6 = \alpha_1$$

$$q_7 = u_2, \quad q_8 = v_2, \quad q_9 = w_2, \quad q_{10} = \gamma_2, \quad q_{11} = -\beta_2, \quad q_{12} = \alpha_2$$





$$\underset{12\times 1}{\{q\}_1} = \underset{12\times 12}{[T_f]_1} \cdot \underset{12\times 1}{\{q_g\}_1}$$

$$\underset{12\times 12}{[T_f]_1} = \left[\begin{array}{cccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & \\
0 & 1 & 0 & 0 & 0 & 0 & \\
0 & 0 & 1 & 0 & 0 & 0 & \underset{6\times 6}{[0]} \\
0 & 0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 0 & -1 & 0 & \\
0 & 0 & 0 & 1 & 0 & 0 & \\
\hline
 & & & & & & \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{matrix} \\
 & & & \underset{6\times 6}{[0]} & & & \\
\end{array}\right] = [T_f]^T$$

$$\underset{12\times12}{[k_g]_1} = \underset{12\times12}{[T_f]_1^T} \cdot \underset{12\times12}{[k]_1} \cdot \underset{12\times12}{[T_f]_1}$$

$$a_1 = \frac{EA_1}{l_1}, \quad b_1 = \frac{12EJ_{z1}}{l_1^3}, \quad c_1 = b_1, \quad d_1 = \frac{6EJ_{z1}}{l_1^2}, \quad e_1 = d_1,$$

$$r_1 = \frac{2EJ_{z1}}{l_1}, \quad s_1 = r_1, \quad t_1 = \frac{GJ_{o1}}{l_1}, \quad A_1 = \frac{\pi\left(d_o^2 - (d_o - 2t)^2\right)}{4}$$

$$J_{z1} = J_{y1} = \frac{\pi}{64}\left(d_o^4 - (d_o - 2t)^4\right), \quad J_{o1} = 2 \cdot J_{z1}$$

$$\underset{18\times18}{[k_g]_1^*} = \left[\begin{array}{c|c} [k_g]_1 & \underset{12\times6}{[0]} \\ \hline \underset{6\times12}{[0]} & \underset{6\times6}{[0]} \end{array}\right]$$

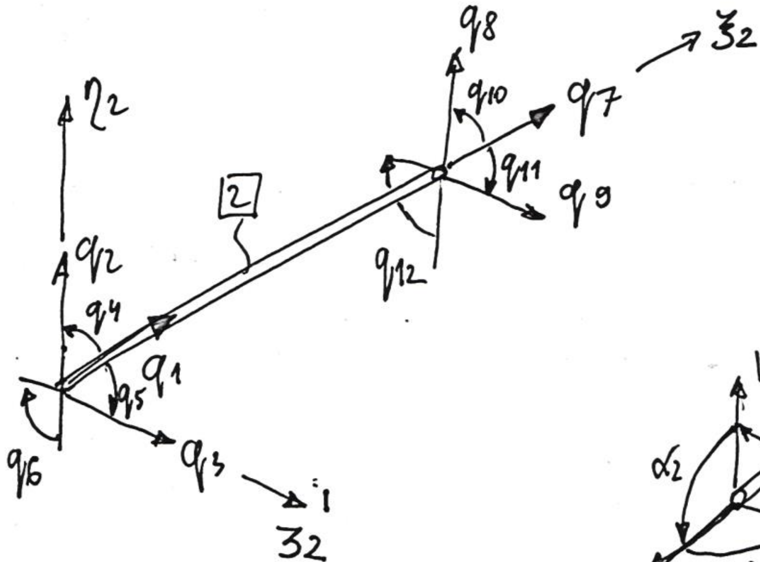
ELEMENT $\boxed{2}$

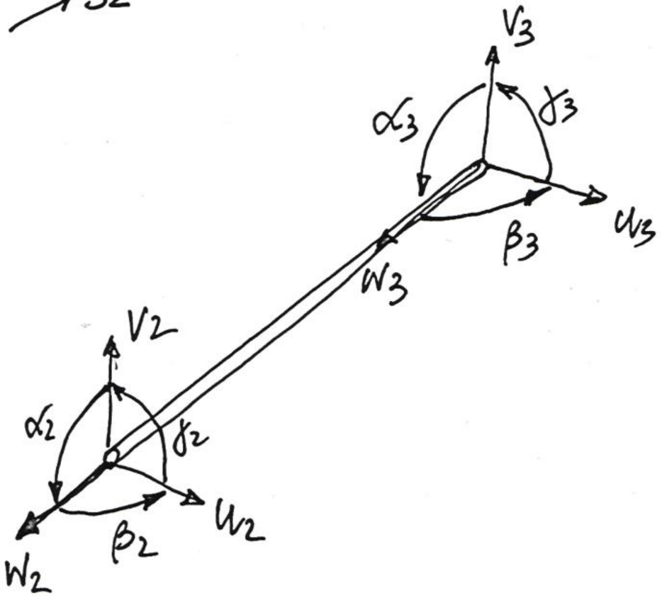
$$\lfloor q \rfloor_2 = \lfloor q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}, q_{11}, q_{12} \rfloor$$

$$\lfloor q_g \rfloor_2 = \lfloor u_2, v_2, w_2, \alpha_2, \beta_2, \gamma_2, u_3, v_3, w_3, \alpha_3, \beta_3, \gamma_3 \rfloor$$

$$q_1 = -w_2, \quad q_2 = v_2, \quad q_3 = u_2, \quad q_4 = \alpha_2, \quad q_5 = -\beta_2, \quad q_6 = -\gamma_2$$

$$q_7 = -w_3, \quad q_8 = v_3, \quad q_9 = u_3, \quad q_{10} = \alpha_3, \quad q_{11} = -\beta_3, \quad q_{12} = -\gamma_3$$





$$\underset{12\times1}{\{q\}_2} = \underset{12\times12}{[T_f]_2} \cdot \underset{12\times1}{\{q_g\}_2}$$

$$[T_f]_2 = \begin{bmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\multicolumn{6}{c}{} & 0 & 0 & -1 & 0 & 0 & 0 \\
\multicolumn{6}{c}{} & 0 & 1 & 0 & 0 & 0 & 0 \\
\multicolumn{6}{c}{[0]} & 1 & 0 & 0 & 0 & 0 & 0 \\
\multicolumn{6}{c}{6\times6} & 0 & 0 & 0 & 1 & 0 & 0 \\
\multicolumn{6}{c}{} & 0 & 0 & 0 & 0 & -1 & 0 \\
\multicolumn{6}{c}{} & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}$$

$$\underset{12\times 12}{[k_g]_2} = \underset{12\times 12}{[T_f]_2^T} \cdot \underset{12\times 12}{[k]_2} \cdot \underset{12\times 12}{[T_f]_2}$$

$$a_2 = \frac{EA_2}{l_2}, \quad b_2 = \frac{12EJ_{32}}{l_2^3}, \quad c_2 = \frac{12EJ_{\eta 2}}{l_2^3}, \quad d_2 = \frac{6EJ_{32}}{l_2^2}$$

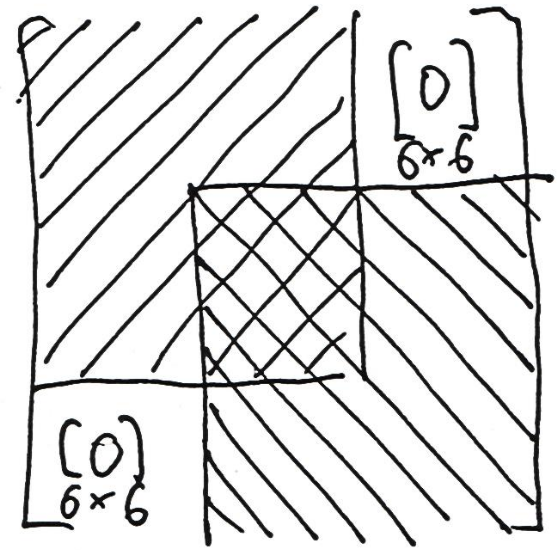
$$e_2 = \frac{6EJ_{\eta 2}}{l_2^2}, \quad r_2 = \frac{2EJ_{32}}{l_2}, \quad s_2 = \frac{2EJ_{\eta 2}}{l_2}, \quad t_2 = \frac{G \cdot J_{02}}{l_2}$$

$$J_{32} = \frac{hb^3}{12}, \quad J_{\eta 2} = \frac{bh^3}{12}, \quad J_{02} = 0.457\, bh^3$$

$$A_2 = b \cdot h$$

$$\underset{18\times 18}{[k_g]_2^*} = \left[\begin{array}{c|c} \underset{6\times 6}{[0]} & \underset{6\times 12}{[0]} \\ \hline \underset{12\times 6}{[0]} & [k_g]_2 \end{array}\right]$$

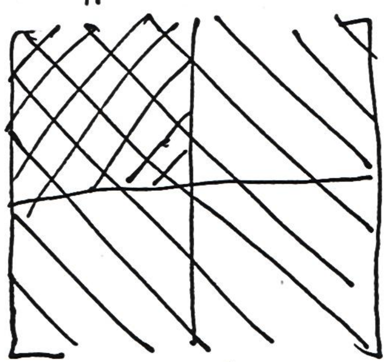
$$\underset{18\times 18}{[K]} = \underset{18\times 18}{[Kg]_1^*} + \underset{18\times 18}{[Kg]_2^*} =$$



$$\underset{18\times 18}{[K]} \cdot \underset{18\times 1}{\{q\}} = \underset{18\times 1}{\{F\}}$$

$$u_1 = 0,\; v_1 = 0,\; w_1 = 0,\; \alpha_1 = 0,\; \beta_1 = 0,\; \gamma_1 = 0$$

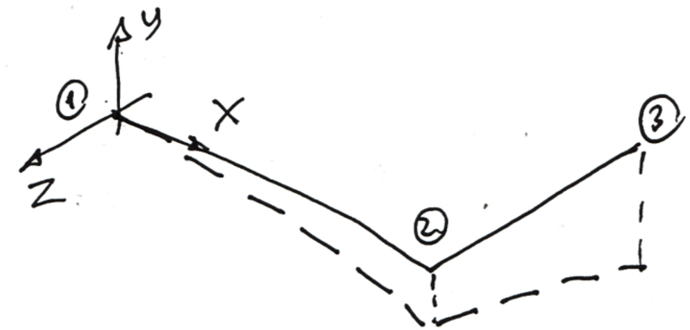
$$\underset{12\times 12}{[K]} \cdot \underset{12\times 1}{\{q\}} = \underset{12\times 1}{\{F\}} \Rightarrow \underset{12\times 1}{\{q\}} = \underset{12\times 12}{[K]^{-1}} \cdot \underset{12\times 1}{\{F\}}$$

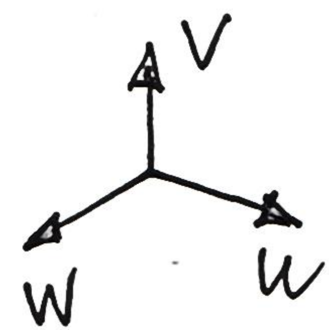


$$\underset{18\times 18}{[K]} \cdot \underset{18\times 1}{\{q\}} = \underset{18\times 1}{\{F\}} \Rightarrow \text{REACTIONS}$$

DOF SOLUTION

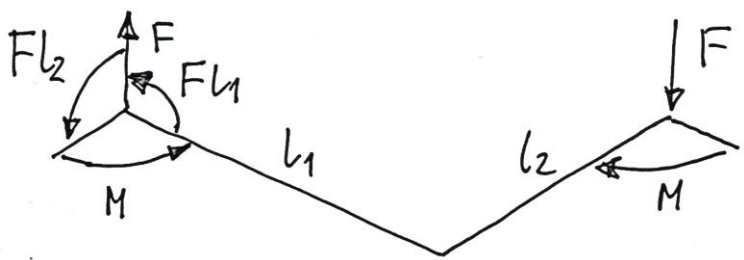
$$\underset{12\times 1}{\{q\}} = \begin{Bmatrix} u_2 \\ v_2 \\ w_2 \\ \alpha_2 \\ \beta_2 \\ \gamma_2 \\ u_3 \\ v_3 \\ w_3 \\ \alpha_3 \\ \beta_3 \\ \gamma_3 \end{Bmatrix} = \begin{Bmatrix} 0 \text{ mm} \\ -11.94 \text{ mm} \\ 14.93 \text{ mm} \\ -0.0243 \text{ rad} \\ -0.0249 \text{ rad} \\ -0.015 \text{ rad} \\ 29.07 \text{ mm} \\ -31.43 \text{ mm} \\ 14.93 \text{ mm} \\ -0.0269 \text{ rad} \\ -0.0527 \text{ rad} \\ -0.015 \text{ rad} \end{Bmatrix}$$





REACTIONS

$$\underset{18\times 1}{\{F\}} = \underset{18\times 18}{[K]} \cdot \underset{18\times 1}{\{q\}} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ M_{RX} \\ M_{RY} \\ M_{RZ} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -F \\ 0 \\ 0 \\ -M \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1000\,N \quad (+F) \\ 0 \\ 750000 \quad (F \cdot l_2) \\ 1000000 \quad (M) \\ 1200000 \quad (F l_1) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -F \\ 0 \\ 0 \\ -M \\ 0 \end{Bmatrix}$$

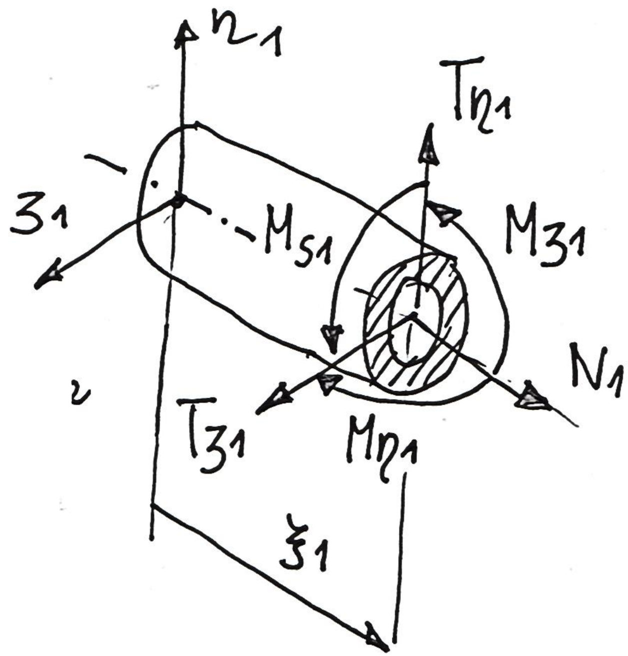


THE STRUCTURE IS IN EQUILIBRIUM

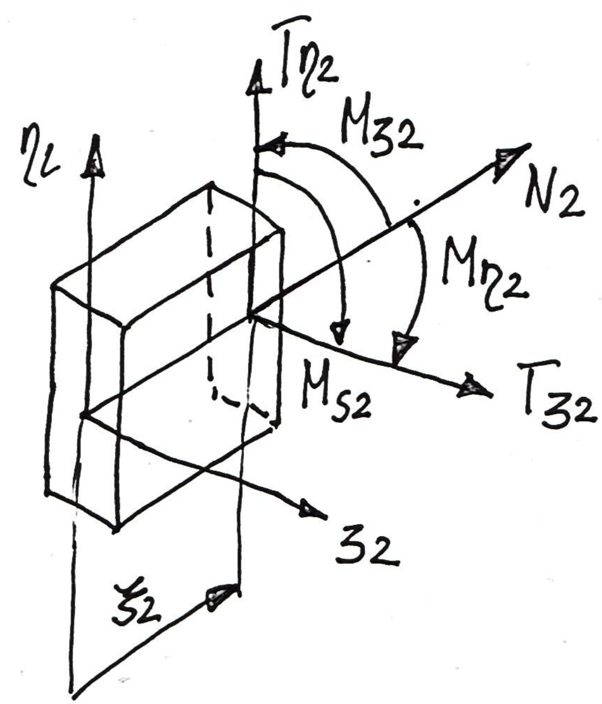
ELEMENT SOLUTION

$$\underset{12\times 1}{\{q\}_i} = \underset{12\times 12}{[T_f]_i} \cdot \underset{12\times 1}{\{q_g\}_i} \quad , \quad i = 1, 2$$

$\boxed{1}$



$\boxed{2}$



AXIAL BAR : $(q_1, q_7)_i$

$$\varepsilon_{\xi i} = \frac{(q_7 - q_1)_i}{l_i} \quad , \quad \sigma_{\xi i} = E \cdot \varepsilon_{\xi i} \quad , \quad N_i = \sigma_{\xi i} A_i$$

BEAM :

I) BENDING IN $(\xi\eta)_i$ PLANE : $(q_2, q_4, q_8, q_{10})_i$

$$v_i(\xi_i) = \underset{1\times 4}{\lfloor N(\xi_i) \rfloor} \cdot \begin{Bmatrix} q_2 \\ q_4 \\ q_8 \\ q_{10} \end{Bmatrix}_i$$

$$M_{zi}(\xi_i) = E \cdot J_{zi} \cdot v_i''(\xi_i) =$$
$$= E \cdot J_{zi} \cdot \underset{1\times 4}{\lfloor N''(\xi_i) \rfloor} \cdot \begin{Bmatrix} q_2 \\ q_4 \\ q_8 \\ q_{10} \end{Bmatrix}_i$$

$$\sigma_{\xi i}^{I} = -\frac{M_{\zeta i}(\xi_i)\cdot \eta_i}{J_{\zeta i}}$$

$$T_{\eta i}(\xi_i) = -EJ_{\zeta i}\cdot v_i'''(\xi_i) =$$

$$= -EJ_{\zeta i}\cdot \underset{1\times 4}{\left[N'''\right]}\cdot \begin{Bmatrix} q_2 \\ q_4 \\ q_8 \\ q_{10} \end{Bmatrix}_i = \text{const}$$

SHEAR STRESS CAUSED BY $T_{\eta i}(\xi_i)$ IS NEGLECTED.

II) BENDING IN $(\xi\zeta)$ PLANE : $(q_3, q_5, q_9, q_{11})_i$

$$w_i(\xi_i) = \underset{1\times 4}{\lfloor N(\xi_i) \rfloor} \cdot \begin{Bmatrix} q_3 \\ q_5 \\ q_9 \\ q_{11} \end{Bmatrix}_i$$

$$M_{\eta i}(\xi_i) = E \cdot J_{\eta i}\, w_i''(\xi_i) =$$

$$= E J_{\eta i} \cdot \underset{1\times 4}{\lfloor N''(\xi_i) \rfloor} \cdot \begin{Bmatrix} q_3 \\ q_5 \\ q_9 \\ q_{11} \end{Bmatrix}_i$$

$$\sigma_{\xi i}^{II} = -\frac{M_{\eta i}(\xi_i)\cdot z_i}{J_{\eta i}}$$

$$T_{z i}(\xi_i) = -E J_{\eta i}\cdot w_i'''(\xi_i) =$$

$$= -E J_{\eta i} \lfloor N''' \rfloor_{1\times 4} \cdot \begin{Bmatrix} q_3 \\ q_5 \\ q_9 \\ q_{11} \end{Bmatrix}_i = \text{cont}$$

SHEAR STRESS CAUSED BY $T_{z i}(\xi_i)$ IS NEGLECTED.

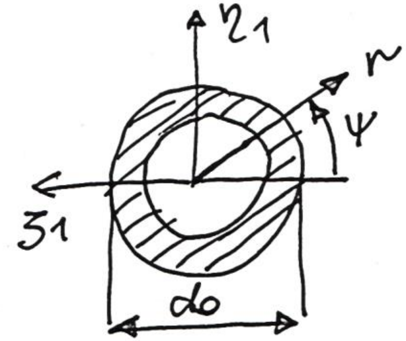
TORSION BAR : $(q_6, q_{12})_i$

$$\varphi_i(\xi_i) = \underset{1\times 2}{\lfloor N(\xi_i) \rfloor} \cdot \begin{Bmatrix} q_6 \\ q_{12} \end{Bmatrix}_i = \left\lfloor 1 - \frac{\xi_i}{l_i} \, , \; \frac{\xi_i}{l_i} \right\rfloor \cdot \begin{Bmatrix} q_6 \\ q_{12} \end{Bmatrix}_i =$$

$$= (q_6)_i + \frac{(q_{12} - q_6)_i}{l_i} \cdot \xi_i$$

$$\begin{Bmatrix} q_6 \\ q_{12} \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

ELEMENT $\boxed{1}$:



$$\tau_1(r) = G \cdot \gamma_1(r) = \frac{E}{2(1+\nu)} \cdot \frac{d\varphi_1(\xi_1)}{d\xi_1} \cdot r =$$

$$= \frac{E}{2(1+\nu)} \cdot \left\lfloor -\frac{1}{L_1} , \frac{1}{L_1} \right\rfloor \cdot \begin{Bmatrix} q_6 \\ q_{12} \end{Bmatrix}_1 \cdot r =$$

$$= \frac{E\,(q_{12} - q_6)_1}{2(1+\nu)\,L_1} \cdot r \quad 12 \quad , (q_6)_1 = 0$$

$$\tau_{1\,max} = \tau_1\left(\frac{d_0}{2}\right) = \frac{E \cdot (q_{12})_1 d_0}{4(1+\nu)\,L_1} = \frac{E\, d_0\, \alpha_2}{4(1+\nu) L_1}$$

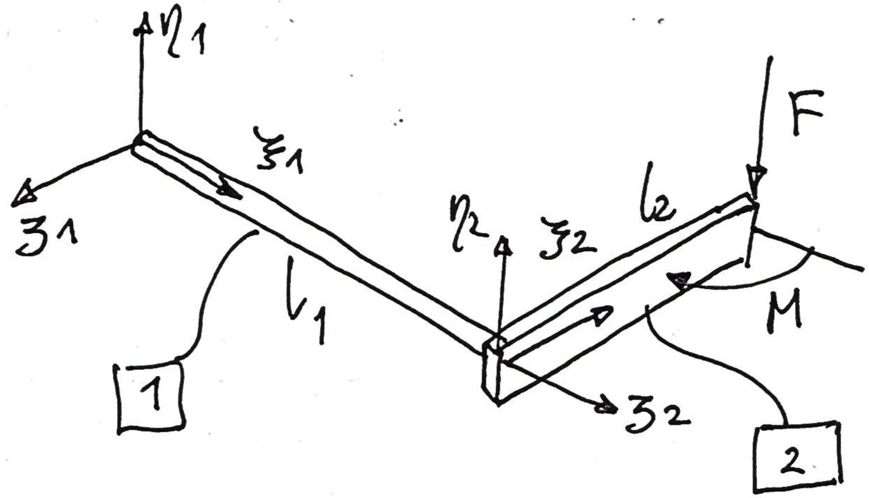
$$M_{s1} = \frac{\tau_1(r) \cdot J_{01}}{r} = \frac{E \cdot (q_{12})_1 \cdot J_{01}}{2(1+\nu)\,L_1} = \frac{E \alpha_2 J_{01}}{2(1+\nu) L_1} = \text{const}$$

ELEMENT $\boxed{2}$:

$$\begin{Bmatrix} q_{6} \\ q_{12} \end{Bmatrix}_2 = \begin{Bmatrix} \gamma_2 \\ \gamma_3 \end{Bmatrix} = \begin{Bmatrix} -0.015 \text{ rad} \\ -0.015 \text{ rad} \end{Bmatrix} \Rightarrow \varphi_2(\xi_2) = q_6 = \text{const}$$

$$\Rightarrow \frac{d\varphi_2(\xi_2)}{d\xi_2} = 0 \Rightarrow \begin{array}{l} \tau_2 = 0 \\ M_{s2} = 0 \end{array}$$

ELEMENT RESULTS



NORMAL FORCES :

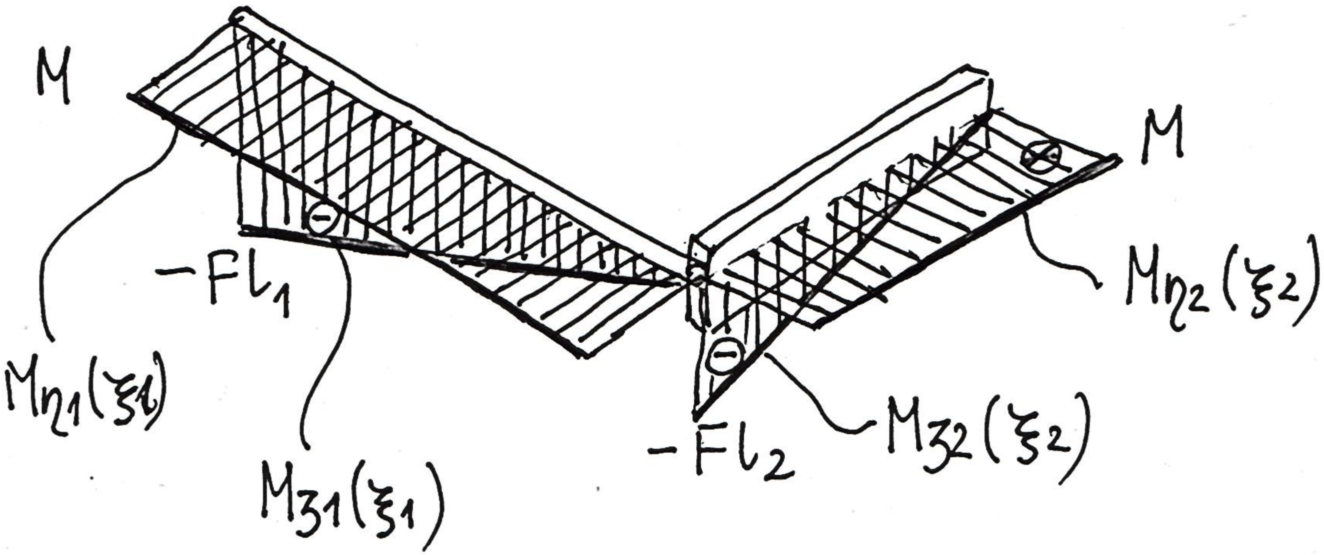
$$N_1 = 0 \quad , \; N_2 = 0$$

SHEAR FORCES :

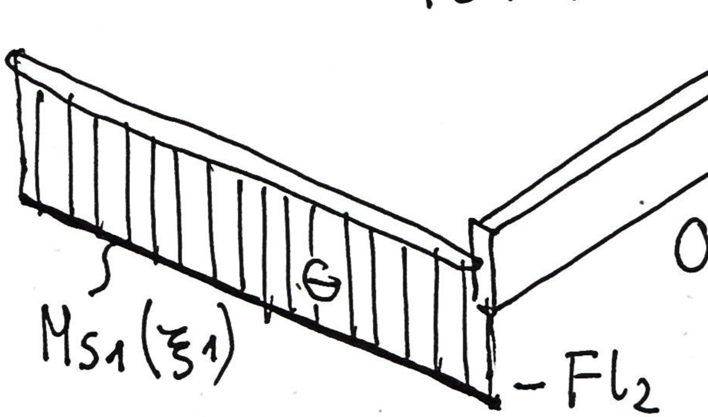
$$T_{\eta_1} = -F \quad , \quad T_{\eta_2} = -F$$

$$T_{\zeta_1} = 0 \quad , \quad T_{\zeta_2} = 0$$

BENDING MOMENTS :



TORQUE. :



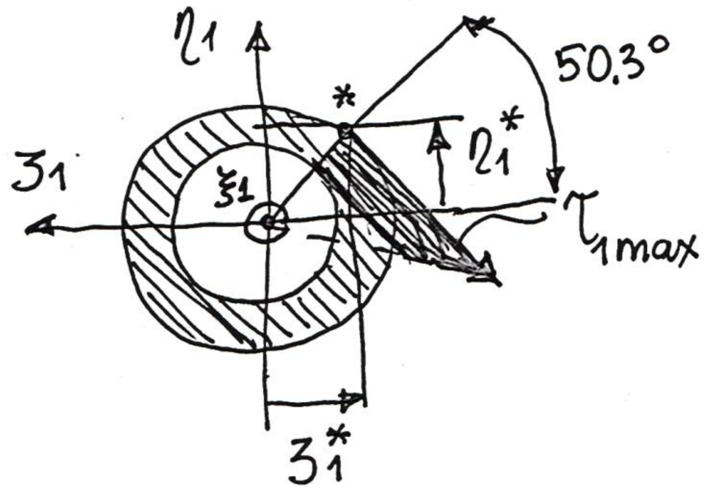
POINT OF THE HIGHEST STRESS:

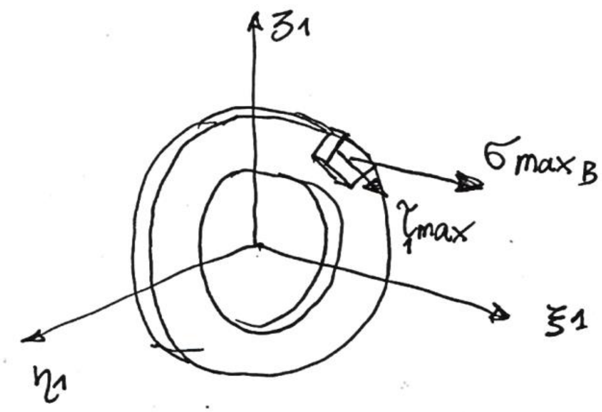
ELEMENT $\boxed{1}$, $\xi_1 = 0$

NORMAL STRESS DUE TO BENDING:

$$\sigma_{MAX\,B} = \sigma_{\xi_1}^{I} + \sigma_{\xi_1}^{II} =$$

$$= -\frac{M_{z1}(0)\cdot \eta_1^*}{J_{z1}} - \frac{M_{\eta 1}(0)\cdot z_1^*}{J_{\eta 1}} = 161.9\,MPa$$





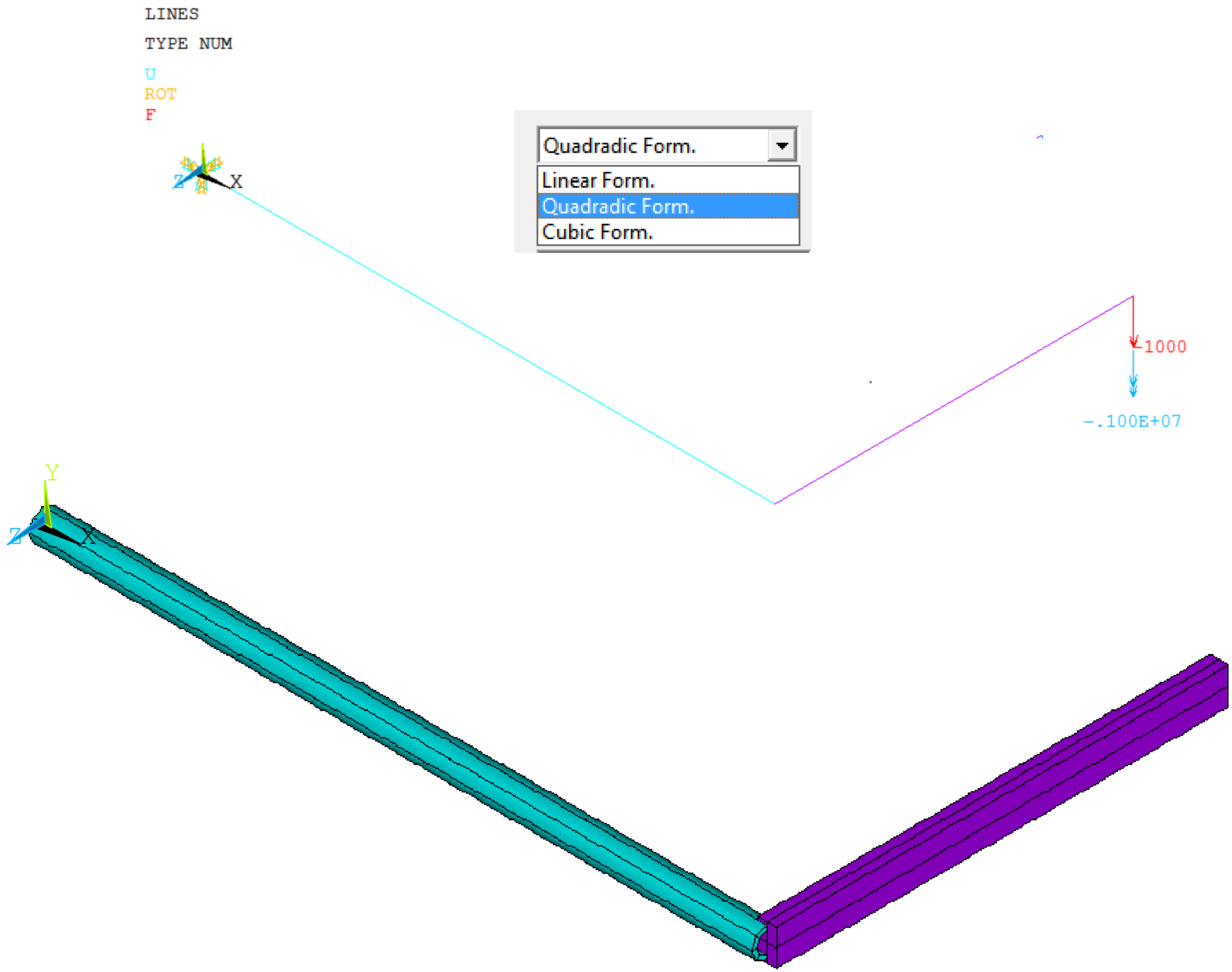
SHEAR STRESS

$$\tau_{1max} = \frac{E d_0 \alpha_2}{4(1+\nu)L_1} = -38.86 \text{ MPa}$$

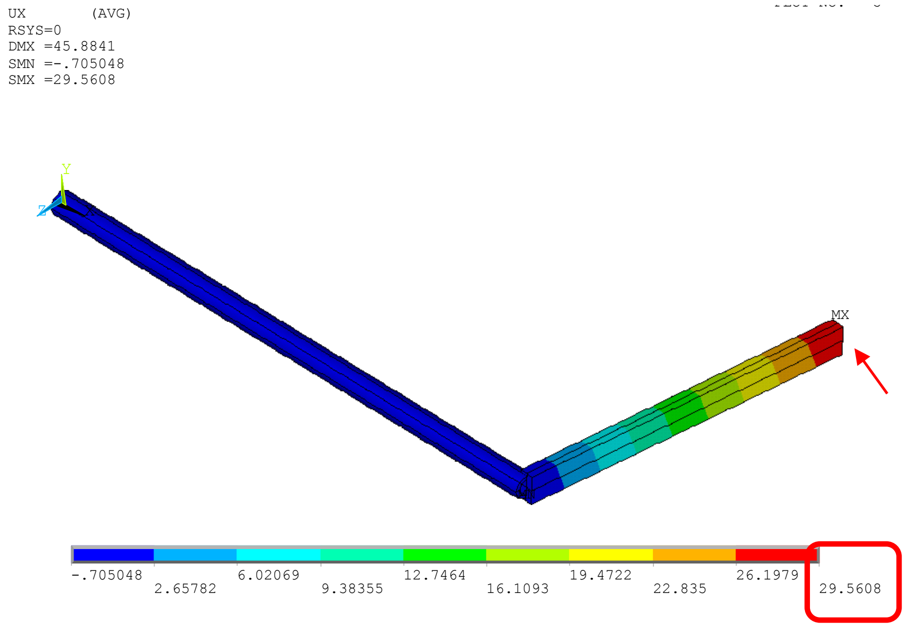
MAXIMUM EQUIVALENT STRESS :

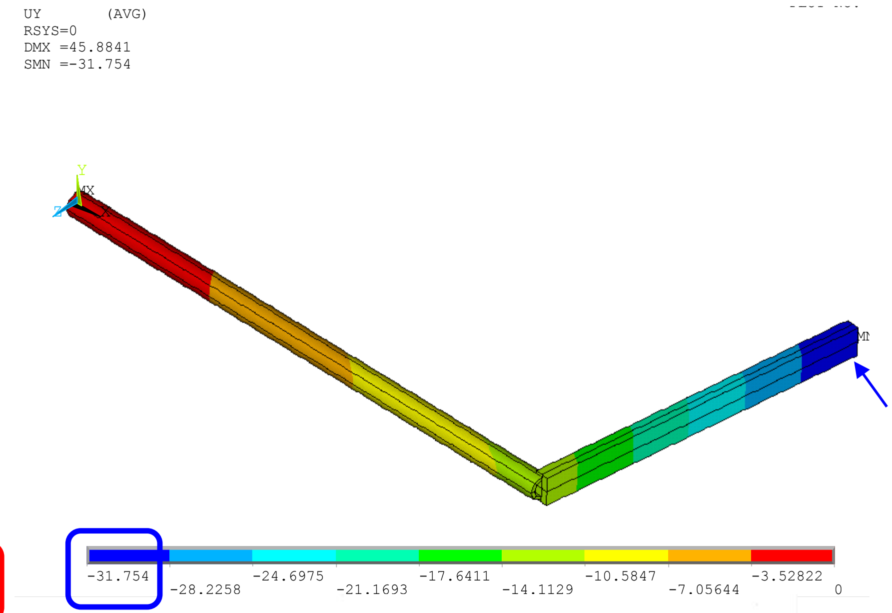
$$\sigma_{EQV} = \sqrt{\sigma_{MAX\,B}^{2} + 3\,\tau_{1max}^{2}} = 175.34 \text{ MPa}$$

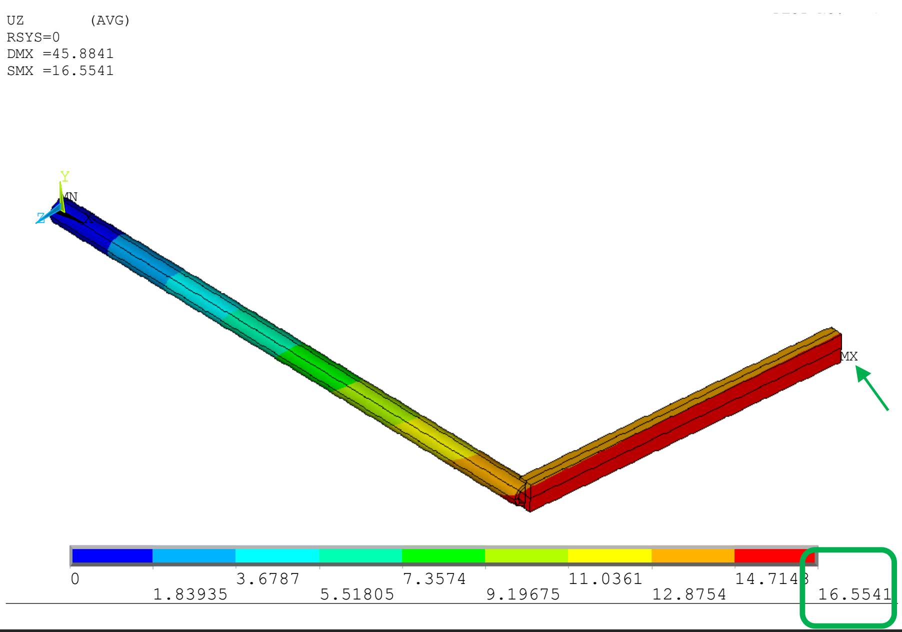
Model dwuelementowy (Ansys)



Model dwuelementowy (Ansys) - Przemieszczenia liniowe

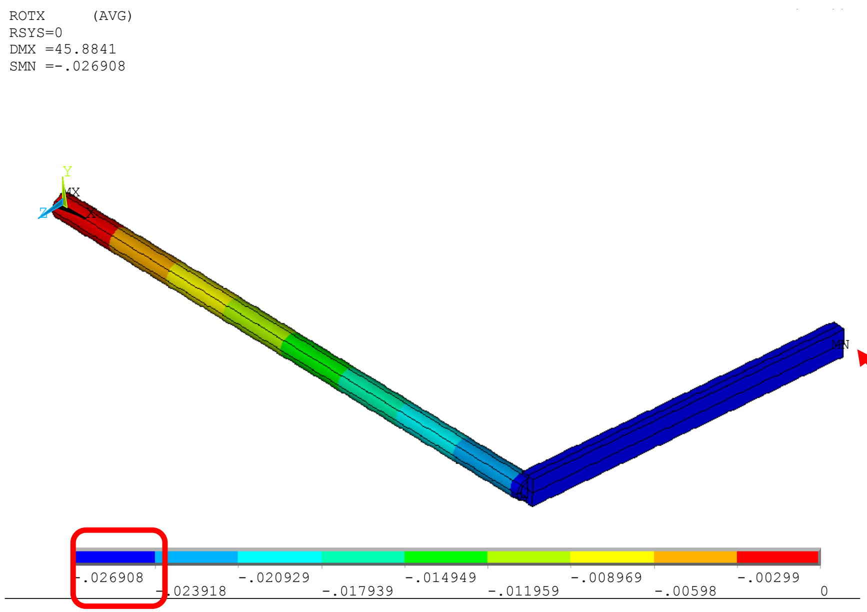


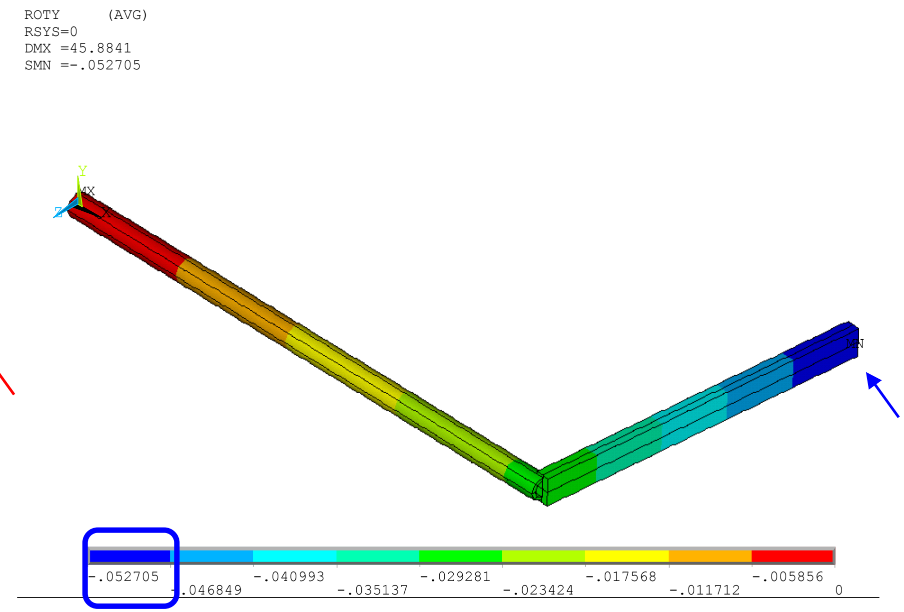


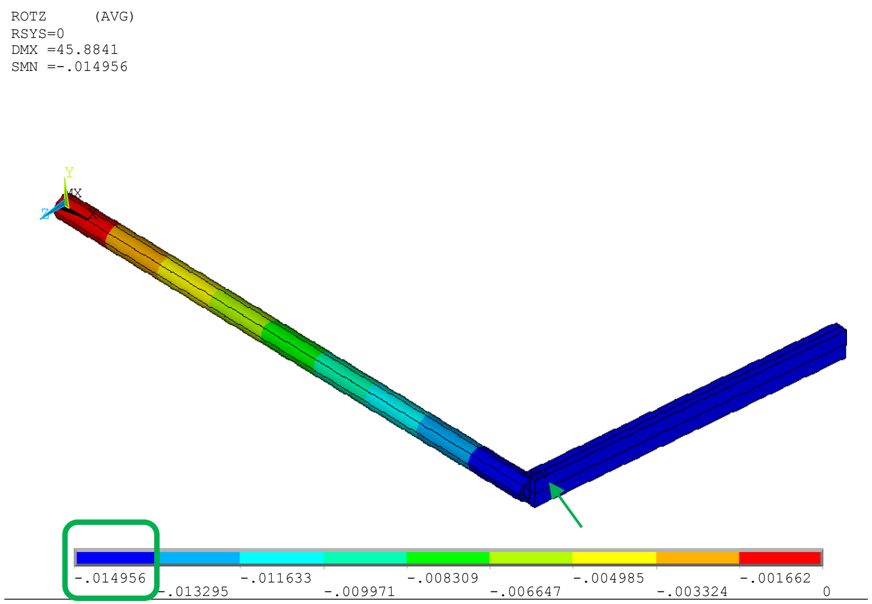


$$\underset{12\times 1}{\{q\}} = \begin{Bmatrix} u_2 \\ v_2 \\ w_2 \\ \alpha_2 \\ \beta_2 \\ \gamma_2 \\ u_3 \\ v_3 \\ w_3 \\ \alpha_3 \\ \beta_3 \\ \gamma_3 \end{Bmatrix} = \begin{Bmatrix} 0\ \text{mm} \\ -11.94\ \text{mm} \\ 14.93\ \text{mm} \\ -0.0243\ \text{rad} \\ -0.0249\ \text{rad} \\ -0.015\ \text{rad} \\ \boxed{29.07\ \text{mm}} \\ \boxed{-31.43\ \text{mm}} \\ \boxed{14.93\ \text{mm}} \\ -0.0269\ \text{rad} \\ -0.0527\ \text{rad} \\ -0.015\ \text{rad} \end{Bmatrix}$$

Model dwuelementowy (Ansys) - Przemieszczenia kątowe

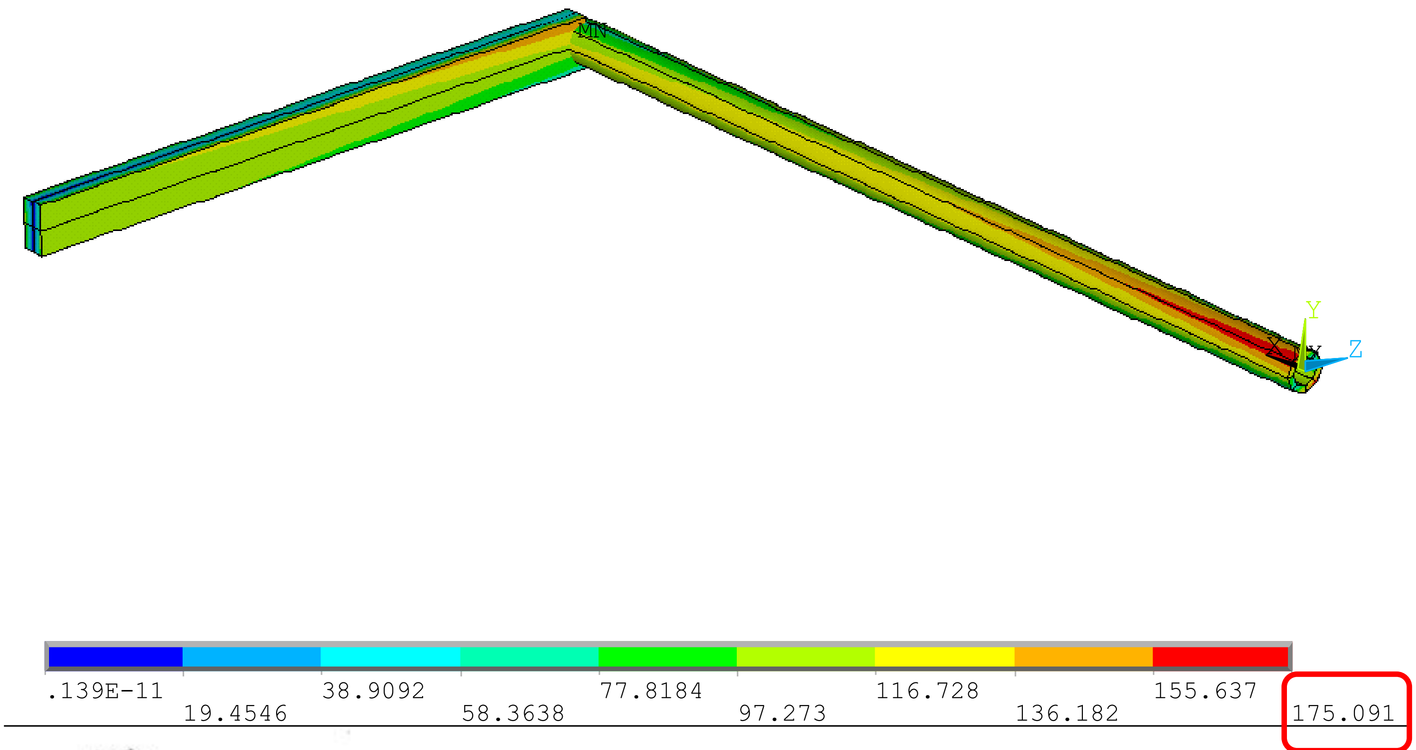


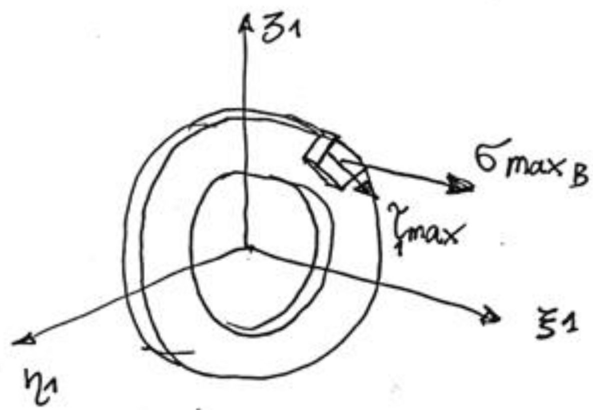




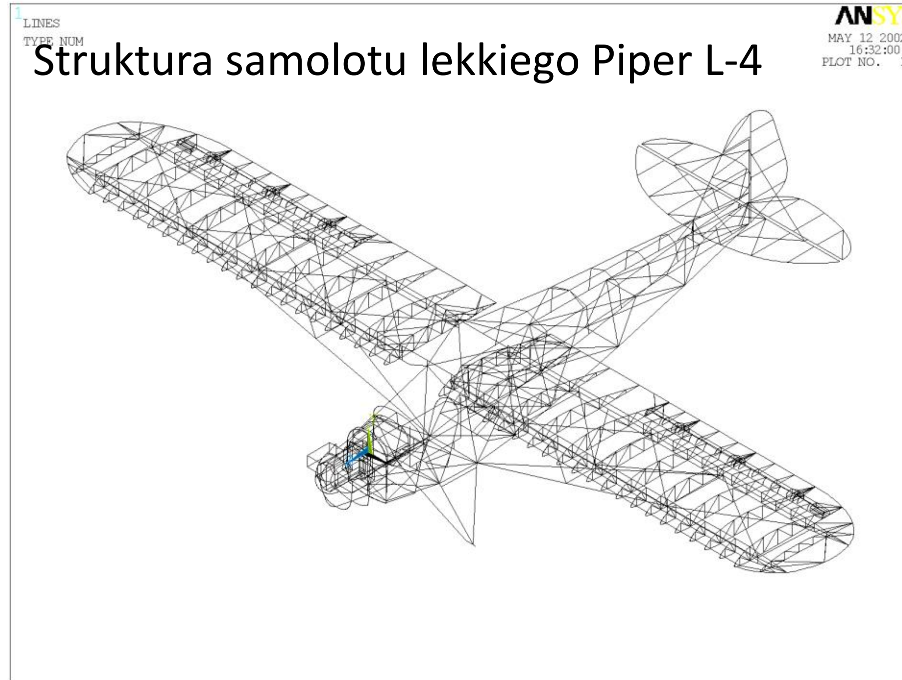
$$\{q\}_{12\times 1} = \begin{Bmatrix} u_2 \\ v_2 \\ w_2 \\ \alpha_2 \\ \beta_2 \\ \gamma_2 \\ u_3 \\ v_3 \\ w_3 \\ \alpha_3 \\ \beta_3 \\ \gamma_3 \end{Bmatrix} = \begin{Bmatrix} 0\ \text{mm} \\ -11.94\ \text{mm} \\ 14.93\ \text{mm} \\ -0.0243\ \text{rad} \\ -0.0249\ \text{rad} \\ \boxed{-0.015\ \text{rad}} \\ 29.07\ \text{mm} \\ -31.43\ \text{mm} \\ 14.93\ \text{mm} \\ \boxed{-0.0269\ \text{rad}} \\ \boxed{-0.0527\ \text{rad}} \\ -0.015\ \text{rad} \end{Bmatrix}$$

Model dwuelementowy (Ansys) - Von Mises stress (SEQV)

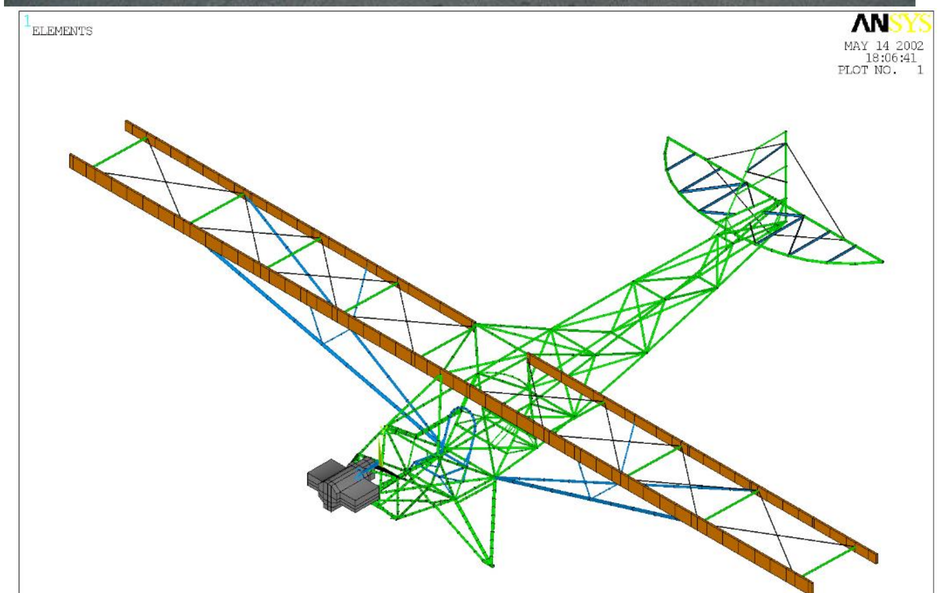




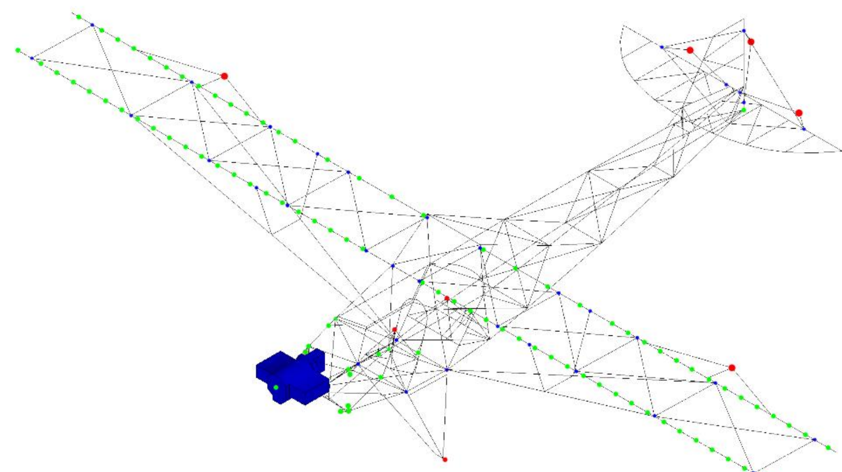
$$\sigma_{EQV} = \sqrt{\sigma_{MAX\,B}^{2} + 3\,\tau_{1\,max}^{2}} = \boxed{175.34\ MPa}$$



Struktura samolotu lekkiego Piper L-4



Model belkowy MES



Rozmieszczenie
mas skupionych
w modelu zastępczym